As discussed in the previous chapter, there are two classical approaches to the analysis of longitudinal data: The first is variably called univariate mixed-model, split-plot, or repeated-measures ANOVA, and the second is based on multivariate ANOVA (MANOVA). In this chapter, we discuss the general multivariate growth curve model. The primary advantage of the MANOVA approach versus the ANOVA approach is that the MANOVA assumes a general form for the correlation of repeated measurements over time, whereas the ANOVA assumes the much more restrictive compound-symmetric form. The disadvantage of the MANOVA model is that it requires complete data. Subjects with incomplete data must be removed from the analysis, leading to potential bias, or have their missing values imputed in some way. In addition, both MANOVA and ANOVA models focus on comparison of group means and provide no information regarding subject-specific growth curves. Finally, both ANOVA and MANOVA models require that the timepoints are fixed across subjects (either evenly or unevenly spaced) and are treated as a classification variable in the ANOVA or MANOVA model. This precludes analysis of unbalanced designs in which different subjects are measured on different occasions. Finally, the MANOVA approach precludes use of time-varying covariates that are often essential to modeling dynamic relationships between predictors and outcomes.

In the following sections, we present the MANOVA model drawing upon the description in Bock [1975]. This text has more details for the interested reader. While this model is no longer recommended for routine application (if at all), it is important in that it helps to fix ideas for the development of the more modern and advanced methods that are the primary focus of this book.
3.1 DATA LAYOUT FOR ANOVA VERSUS MANOVA

A primary distinction between application of the ANOVA and MANOVA models for repeated measures concerns the data arrangement. For the ANOVA approach, each subject's data across \( n \) timepoints consists of \( n \) different observations in the dataset. For example, for a study with three timepoints, we might have the following lines for the first two subjects:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Time</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( y_{11} )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( y_{12} )</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( y_{13} )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( y_{21} )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( y_{22} )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( y_{23} )</td>
</tr>
</tbody>
</table>

Here, subject 1 is observed at three timepoints (coded 1, 2, and 3), and the values of the time-varying dependent variable are given on consecutive lines. There is one dependent variable \( y \) and two variables indicating subjects and time. This data layout is sometimes called the "univariate setup." Of course, there might be more variables in the datafile, but the essential point is that the repeated measures are considered to be multiple instances of the one dependent variable \( y \).

Alternatively, in the "multivariate setup," used in MANOVA, the dependent variable across \( n \) timepoints consists of \( n \) different variables. Continuing with the same example, we would have:

<table>
<thead>
<tr>
<th>Subject</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y_{11} )</td>
<td>( y_{21} )</td>
<td>( y_{31} )</td>
</tr>
<tr>
<td>2</td>
<td>( y_{12} )</td>
<td>( y_{22} )</td>
<td>( y_{32} )</td>
</tr>
</tbody>
</table>

Notice that the time variable is not explicitly provided, though it is implicit in the number of repeated measures (e.g., here it is three for \( y_1, y_2, \) and \( y_3 \)). This data setup provides a clue as to why MANOVA does not include subjects with incomplete data across time. Essentially, MANOVA treats the repeated measures as one data vector (i.e., one multivariate observation of the dependent variable), and the entire data vector must be complete for the subject to be included in the analysis.

Because these two models require different data setups, it is useful to be able to translate datasets between these two formats. Below is SAS code, adapted from Littell et al. [1991], for reading in a dataset in multivariate form and translating it into univariate form. This code reads in two subjects with dependent variable values of 1, 2, and 3 for the first subject, and values of 4, 5, and 6 for the second subject. All SAS-specific syntax is indicated in upper-case letters, optional user-specified names are indicated in lower case. The multivariate dataset, named multdat, contains three variables: \( y_1, y_2, \) and \( y_3 \). The univariate dataset, named unidat, contains three variables subject, time, and \( y \). The code below utilizes an ARRAY and a DO loop to convert the data to the univariate setup.
The array yv is designated to contain the three variables $y_1$, $y_2$, and $y_3$. The subject assignment statement creates the subject variable which is incremented (by 1) for each subsequent line in the multivariate dataset. The DO loop creates the time variable, taking on values of 1, 2, and 3. The assignment statement, within the DO loop, assigns the values of the variables $y_1$, $y_2$, and $y_3$ to the univariate $y$. Following each assignment in the loop, the OUTPUT statement creates a new observation.

```sas
DATA multdat;
INPUT y1 y2 y3;
DATALINES;
1 2 3
4 5 6
;
DATA unidat;
SET multdat;
ARRAY yv(3) y1 y2 y3;
subject + 1;
DO time = 1 TO 3;
   y = yv(time);
   OUTPUT;
END;
DROP y1 y2 y3;
RUN;
```

To go in the opposite direction, the SAS code below reads in the same dataset in univariate form and then translates it to multivariate form. Again, an ARRAY and a DO loop are used in the conversion. Note the different placement of the SET statement below, relative to the code above, and also the difference in the statement, within the DO loop, assigning the dependent variable values of the univariate $y$ to the multivariate $y_1$, $y_2$, and $y_3$.

```sas
DATA unidat;
INPUT subject time y;
DATALINES;
1 1 1
1 2 2
1 3 3
2 1 4
2 2 5
2 3 6
;
DATA multdat;
ARRAY yv(3) y1 y2 y3;
DO time = 1 TO 3;
   SET unidat;
   yv(time) = y;
END;
DROP y subject time;
RUN;
```
3.2 MANOVA FOR REPEATED MEASUREMENTS

In the MANOVA approach to analysis of repeated measurements, the \( n \) repeated measures are treated as a \( n \times 1 \) response vector \( y_i \). Due to the multivariate nature of the analysis, subjects with any missing \( y_{ij} \) (across time) are omitted from the analysis. This is the "Achilles heel" of the MANOVA model for repeated measurements and why it is largely of only limited use in many research fields. The one-sample MANOVA model is given by

\[
y_i = \mu + \varepsilon_i,
\]

where \( \mu = n \times 1 \) mean vector for timepoints, and \( \varepsilon_i = n \times 1 \) vector of errors, distributed as \( N(0, \Sigma) \) in the population. Unlike the univariate repeated measures model of the last chapter, this specification indicates that the error variance-covariance matrix is allowed to be completely general. For the one-sample case, we can characterize the timepoint vector \( \mu \) and choose contrasts depending on the structure and hypotheses of interest. Below, we will present the model using orthogonal polynomial contrasts, but other sets of contrasts, like those described in the previous chapter, can be used instead.

Notice that under the univariate model assumptions, the variance-covariance matrix is \( \Sigma = \sigma^2 \mathbf{I}_n + \sigma^2 \mathbf{I}_n \) and the mean vector is \( \mu = \mu + \tau \), which are the grand mean and time effects in the univariate model notation. As a result, as we shall see, all univariate results can be extracted from the multivariate model.

3.2.1 Growth Curve Analysis—Polynomial Representation

The mean vector of the polynomial growth-curve model can be characterized as

\[
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_n
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix} \beta_0 +
\begin{bmatrix}
t_1 \\
t_2 \\
\vdots \\
t_n
\end{bmatrix} \beta_1 +
\begin{bmatrix}
t_1^2 \\
t_2^2 \\
\vdots \\
t_n^2
\end{bmatrix} \beta_2 + \ldots +
\begin{bmatrix}
t_1^{q-1} \\
t_2^{q-1} \\
\vdots \\
t_n^{q-1}
\end{bmatrix} \beta_{q-1},
\]

where \( t_1, t_2, \ldots, t_n \) represent timepoint values, and \( q \leq n \) represents the degree of the polynomial. It is generally advantageous to orthogonalize \( T \) as \( \mu = P' \theta \), where \( P \) is the \( q \times n \) matrix of orthogonal polynomials. The first row of \( P \) is for the constant term, and the remaining rows correspond sequentially to the linear, quadratic, etc. Note that \( P = S^{-1} T \) and \( S S' = (T T') \), which is the Cholesky factorization, where \( S \) is a \( q \times q \) lower triangular matrix. For equal time intervals, these orthogonal polynomial contrasts can be found in several statistics texts—for example, in Pearson and Hartley [1976].

Alternatively, the SAS PROC IML statements below can be used to produce \( P \) based on \( T \), here considering four timepoints. Note that in the code below, \( T \) is named time and \( P \) is named orthpoly.
TITLE 'producing orthogonal polynomial matrix';
PROC IML;
  time = { 1 1 1 1 ,
            0 1 2 3 ,
            0 1 4 9 ,
            0 1 8 27 };  
  orth poly = t(inv(root(time*t(time)))*time);
PRINT 'time matrix', time [FORMAT=8.4];
PRINT 'orthogonalized time matrix', orth poly [FORMAT=8.4];

This code uses several built-in SAS matrix routines: \text{T} which obtains the transpose of a matrix, \text{INV} which performs matrix inversion, and \text{ROOT} which yields the transpose of the Cholesky factor (i.e., \text{ROOT} yields the upper triangular matrix $S'$). Additionally, matrix multiplication is performed using the \ast operator, which is also the ordinary scalar multiplication operator.

Using either the tabled contrast values, or those obtained using the SAS code above, yields the following values for $q = n = 4$:

$$P_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-3 & -1 & 1 & 3 \\
1 & -1 & -1 & 1 \\
-1 & 3 & -3 & 1
\end{bmatrix} \div \sqrt{4} \ p_0, \frac{\sqrt{20}}{4} \ p_1, \frac{\sqrt{4}}{4} \ p_2, \frac{\sqrt{4}}{4} \ p_3.$$  

The division sign to the right of the matrix indicates that the elements of each row are to be divided by the indicated square root quantity for that row. These quantities are simply the sum of squares of the row elements, and so dividing the elements of the matrix by these yields polynomial contrasts (i.e., $p_0$ = constant, $p_1$ = linear, $p_2$ = quadratic, and $p_3$ = cubic) that are on the same scale, and so can be more directly compared to each other. The orthogonal polynomial trend model is therefore

$$Py_i = P\mu + P\varepsilon_i$$

$$\quad = \theta + \varepsilon_i^*,$$

where

- $\theta = n \times 1$ vector of transformed population means with its least squares estimate given by the transformed sample mean vector, namely $\hat{\theta} = P\hat{y}$, (where $\hat{y}$ is the $n \times 1$ vector of timepoint means), and

- $\varepsilon_i^* \sim N(0, \Sigma^* = PP^t)$.  

Note that the univariate ANOVA of the last chapter assumes that $PS^tP'$ is diagonal with equal values below the first element (since $P$, as defined above, includes the zero-order term, i.e., the grand mean, as the first row). Specifically, the test of sphericity examines whether the lower $(n-1) \times (n-1)$ partition of the $n \times n$ matrix $PS^tP'$ equals a constant on the diagonal and zero for all off-diagonal elements.
The MANOVA table, with SSCP denoting a sum of squares and cross-product matrix (or more simply a cross-product matrix), is given as

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SSCP ((n \times n))</th>
<th>E(SSCP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>1</td>
<td>(N P \bar{y} \bar{y}' P')</td>
<td>(P [\Sigma + N \mu \mu'] P')</td>
</tr>
<tr>
<td>Residual</td>
<td>(N - 1)</td>
<td>(SSR^* = P SSR P')</td>
<td>((N - 1) P \Sigma P')</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= P(Y'Y - N \bar{y} \bar{y}') P')</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(N)</td>
<td>(SSY^* = PY'YP')</td>
<td></td>
</tr>
</tbody>
</table>

In the above table, \(Y\) is the \(N \times n\) matrix of all data, and \(\bar{y}\) is the \(n \times 1\) vector of timepoint means. The degrees of freedom for each \(n \times n\) cross-product matrix reflects the amount of between-subjects information. Based on the above formulation, \(SST^*\) has as its first diagonal element \(Nn\bar{y}_p^2\), which is a function of the grand mean, while the remaining \(n - 1\) diagonal elements are the orthogonal polynomial decomposition of the Time SS (i.e., \(N \sum_{j=1}^{n}(\bar{y}_j - \bar{y})^2\)). \(SSR^*\) has its first diagonal element as the subjects SS (i.e., \(n \sum_{i=1}^{N}(\bar{y}_i - \bar{y})^2\)), with the other \(n - 1\) diagonal elements as the orthogonal polynomial decomposition of Error (i.e., Subject by Time) SS.

The orthogonal polynomial partition of sum of squares and products is

\[
SST^* = \begin{bmatrix}
\text{constant} \\
\text{linear time} \\
\text{quadratic time} \\
\vdots \\
(n - 1)\text{th time}
\end{bmatrix}
\]

\[
SSR^* = \begin{bmatrix}
\text{subjects} \\
\text{subjects \times linear} \\
\text{subjects \times quadratic} \\
\vdots \\
\text{subjects \times } (n - 1)
\end{bmatrix}
\]

Note that both of these are symmetric matrices, and they contain all of the information that is necessary either for extracting the univariate repeated measures ANOVA results (if sphericity is satisfied) or for the more general MANOVA model. Both approaches will now be described.
## 3.2.2 Extracting Univariate Repeated Measures ANOVA Results

If sphericity holds, then the univariate repeated measures results can be extracted directly from the SST* and SSR* matrices as follows:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>E(PS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subjects</td>
<td>$N - 1$</td>
<td>$SS_S = n \sum_{i=1}^{N} (\bar{y}<em>i - \bar{y}</em>..)^2$</td>
<td>$\frac{SS_S}{N-1}$</td>
<td>$\sigma^2_e + n\sigma^2_\tau$</td>
</tr>
<tr>
<td>Time</td>
<td>$n - 1$</td>
<td>$SS_T = N \sum_{j=1}^{n} (\bar{y}<em>{j} - \bar{y}</em>..)^2$</td>
<td>$\frac{SS_T}{n-1}$</td>
<td>$\sigma^2_e + \frac{1}{N} \sum_{j}(\tau_j - \tau.)^2$</td>
</tr>
<tr>
<td>Residual</td>
<td>$(N-1)$</td>
<td>$SS_R = \sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i - \bar{y}<em>j + \bar{y}</em>..)^2$</td>
<td>$\frac{SS_R}{(N-1)(n-1)}$</td>
<td>$\sigma^2_e$</td>
</tr>
<tr>
<td>Total</td>
<td>$Nn - 1$</td>
<td>$SS_y = \sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \bar{y}_..)^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where

$SS_S = sst_0$ from the SSR* matrix,

$SS_T = \text{sum of the lower } n - 1 \text{ diagonal elements of SST*}$, and

$SS_R = \text{sum of the lower } n - 1 \text{ diagonal elements of SSR*}$.

Thus, these SS quantities are easily obtained from the SST* and SSR* matrices. For the $F$-tests, the MS quantities for Subjects and Time are divided by $MS_R$ in the usual way. In terms of the residual mean square, notice that

$$MS_R = \frac{\text{sum of the lower } n - 1 \text{ diagonal elements of SSR*}}{(N - 1)(n - 1)} = \frac{\text{average of the } n - 1 \text{ SSR* diagonal elements}}{(N - 1)}.$$

As a result, the univariate $F$-test for Time, which use $MS_R$ as the denominator, is sometimes referred to as an "averaged" test because of this averaging across the the $n - 1$ elements of the SSR* matrix to yield a single error term. Notice that sphericity maintains that these $n - 1$ elements are equal and uncorrelated, which justifies the averaging of these error variance elements. As a result of this averaging, the denominator degrees of freedom are greater for the univariate $F$-test of Time, relative to the MANOVA test which is subsequently described. This results in greater power for the univariate test, relative to the multivariate test, and is the primary reason why, if sphericity holds, there is generally an advantage to using the univariate repeated measures ANOVA model rather than the MANOVA model.
### 3.2.3 Multivariate Test of the Time Effect

In order to test the null hypothesis of no effect of time, *i.e.*, \( H_0 : \mu \) elements are all equal such that \( H_0 : \tau = 0 \), we must extract and compare the lower \((n - 1) \times (n - 1)\) submatrices of \( \text{SST}^* \) and \( \text{SSR}^* \). As indicated in the MANOVA table, these two submatrices have the same expectation if the null hypothesis is true. Thus, to the extent that there is a time effect, and the null hypothesis is not true, the time \( \text{SST}^* \) submatrix will contain larger elements than the residual \( \text{SSR}^* \) submatrix. Thus, the logic here is similar to a univariate \( F \)-test where we compare two mean squares (*i.e.*, \( \text{MS}_T / \text{MS}_R \)) to test the null hypothesis. Here, of course, we cannot form a simple ratio because we are dealing with matrices. Instead, to compare them, we can solve the determinantal equation:

\[
| \text{SST}_{(n-1)}^* - \lambda \text{SSR}_{(n-1)}^* | = 0
\]  

which has one nonzero latent root or eigenvalue \( \lambda_1 \). Note that this latent root equals one if \( \text{SST}_{(n-1)}^* = \text{SSR}_{(n-1)}^* \). Thus, to the extent that the null hypothesis is true, the latent root will be approximately equal to one.

The above equation is a two-matrix eigenproblem. It can be simplified to a one-matrix eigenproblem, or characteristic equation, using the Cholesky factorization, \( \text{SST}_{(n-1)}^* = \mathbf{E} \mathbf{E}' \), to yield

\[
| \mathbf{E}^{-1} \text{SST}_{(n-1)}^* (\mathbf{E}^{-1})' - \lambda \mathbf{I}_{(n-1)} | = 0.
\]  

Overall test statistics for the null hypothesis of no time effect include Roy's largest root statistic (the latent root or eigenvalue \( \lambda_1 \)), and Wilk's Lambda \( (\Lambda = 1/(1 + \lambda_1)) \). Functions of these test statistics approximately follow an \( F \)-distribution (under the null hypothesis), though sometimes interpolation is necessary, giving rise to fractional df in some cases. Other multivariate test statistics include the Hotelling-Lawley trace and the Pillai-Bartlett trace.

### 3.2.4 Tests of Specific Time Elements

To test specific components of the time effect, there are two options depending on whether sphericity is reasonable or not. First, if sphericity is reasonable and the univariate repeated measures ANOVA is used, then we can obtain univariate test statistics by extracting the lower \( n - 1 \) diagonal elements of \( \text{SST}^* \), with \( \text{MS}_R \) as a common denominator for all trend components, *i.e.,*

\[
F_1 = \frac{\text{SST}_1}{\sum_{j=1}^{n-1} \text{SSR}_j / [(n-1)(N-1)]} \quad \text{linear},
\]

\[
F_2 = \frac{\text{SST}_2}{\sum_{j=1}^{n-1} \text{SSR}_j / [(n-1)(N-1)]} \quad \text{quadratic},
\]

\[
\vdots
\]

\[
F_{n-1} = \frac{\text{SST}_{n-1}}{\sum_{j=1}^{n-1} \text{SSR}_j / [(n-1)(N-1)]} \quad (n-1)\text{th}.
\]  

(3.6)
Note that the summation in the denominator is over the lower \( n - 1 \) diagonal elements of the \( \text{SSR}^* \) matrix (i.e., it does not include \( \text{ssr}_0 \)). Again, this denominator is akin to an averaged estimate of the error variance across time, and each of these \( F \)-statistics is evaluated with 1 numerator and \((n - 1)(N - 1)\) denominator degrees of freedom.

If sphericity is rejected and a MANOVA is performed, then the aforementioned multivariate test of the overall time effect, extracting and comparing the lower \( n - 1 \) diagonal elements of \( \text{SSM}^* \) and \( \text{SSR}^* \), is followed up with the following univariate \( F \)-tests of specific trend components:

\[
F_1 = \frac{\text{SST}_1}{\text{ssr}_1/(N-1)} \quad \text{linear,}
\]

\[
F_2 = \frac{\text{SST}_2}{\text{ssr}_2/(N-1)} \quad \text{quadratic},
\]

\[
\vdots \quad \vdots
\]

\[
F_{n-1} = \frac{\text{SST}_{n-1}}{\text{ssr}_{n-1}/(N-1)} \quad (n - 1)\text{th}.
\]

Here, there is no pooling across time to obtain an averaged error term. Instead, each trend component has its own error term from the \( \text{SSR}^* \) matrix, and so these \( F \)-statistics are evaluated with 1 numerator and only \((N - 1)\) denominator degrees of freedom. This difference in the denominator degrees of freedom is why, in general, the univariate repeated measures model tests (i.e., assuming sphericity) of the trend components in (3.6), which use the pooled error term, are more powerful than the corresponding tests under the MANOVA model in (3.7).

### 3.3 MANOVA OF REPEATED MEASURES—S SAMPLE CASE

In the case of multiple groups, let \( h = 1, \ldots, s \) groups, \( i = 1, \ldots, N_h \) subjects in group \( h, j = 1, \ldots, n \) timepoints, and \( N = \sum N_h \) total number of subjects. Thus, the number of subjects per group can vary, but each subject is measured at \( n \) timepoints. The model is written as

\[
y_{hi} = \mu + \gamma_h + \epsilon_{hi}
\]

where

\( \mu \) is the \( n \times 1 \) vector of timepoint means,

\( \gamma_h \) is the \( n \times 1 \) vector effect for the population from which the \( h \)th group of subjects was drawn,

\( \epsilon_{hi} \) is the \( n \times 1 \) vector of errors distributed as \( N(0, \Sigma) \) in each of the populations.
The model assumes homogeneity of variance–covariance across the \( s \) groups (i.e., the same general error variance–covariance matrix \( \Sigma \) for all \( s \) groups). Again, with orthogonal transformation for the time effects, the model is written as

\[
P\gamma_{hi} = P\mu + P\gamma_h + P\epsilon_{hi},
\]

with \( \epsilon_{hi} \sim N(0, \Sigma^*) = P \Sigma P' \). As in the one-sample case, we can test \( \Sigma^* \) for sphericity and proceed using univariate “averaged” tests if appropriate, or the tests of the MANOVA model if sphericity is violated. The resulting MANOVA table is given by

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SSCP ((n \times n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>1</td>
<td>( \text{SST}^* = P \text{SST} P' = NP\tilde{y}, \tilde{y}', P' )</td>
</tr>
<tr>
<td>Group</td>
<td>( s-1 )</td>
<td>( \text{SSG}^* = P \text{SSG} P' = P(\sum_h N_h \tilde{y}_h, \tilde{y}_h' - \text{SST}) P' )</td>
</tr>
<tr>
<td>Residual</td>
<td>( N-s )</td>
<td>( \text{SSR}^* = P \text{SSR} P' = P(\text{SSY} - \text{SSG} - \text{SST}) P' )</td>
</tr>
<tr>
<td>Total</td>
<td>( N = \sum N_h )</td>
<td>( \text{SSY}^* = P \text{SSY} P' = P(\sum_h \sum_i y_{hi} y_{hi}' P' )</td>
</tr>
</tbody>
</table>

where the degrees of freedom for each \( n \times n \) cross-product matrix reflects the amount of between-subjects information. Note that the results only depend on the following summary statistics: (a) the cross-product matrix from the overall mean vector of the repeated measures \( \tilde{y}, \tilde{y}' \), (b) the sum of cross-product matrices from the group mean vectors of the repeated measures \( \sum_h N_h \tilde{y}_h, \tilde{y}_h' \), and (c) the sum of cross-product matrices from the subject data vectors of the repeated measures \( \sum_h \sum_i y_{hi} y_{hi}' \).

Under the present orthogonal polynomial parameterization of the model, the information in the cross-product matrices is as follows.

<table>
<thead>
<tr>
<th>diagonal element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time: df = 1</td>
</tr>
</tbody>
</table>
| \( \text{SST}^* =  
\begin{bmatrix}
  \text{sst}_0 \\
  \text{sst}_1 \\
  \text{sst}_2 \\
  \cdots \\
  \text{sst}_{n-1}
\end{bmatrix} 
\) |
| constant         |
| linear time      |
| quadratic time   |
| \( (n-1) \text{th time} \) |

| Between groups: df = \( s-1 \) |
| \( \text{SSG}^* =  
\begin{bmatrix}
  \text{ssg}_0 \\
  \text{ssg}_1 \\
  \text{ssg}_2 \\
  \cdots \\
  \text{ssg}_{n-1}
\end{bmatrix} 
\) |
| groups            |
| groups \times linear |
| groups \times quadratic |
| \( \text{groups} \times (n-1) \text{th time} \) |

| Subjects within groups: df = \( N-s \) |
| \( \text{SSR}^* =  
\begin{bmatrix}
  \text{ssr}_0 \\
  \text{ssr}_1 \\
  \text{ssr}_2 \\
  \cdots \\
  \text{ssr}_{n-1}
\end{bmatrix} 
\) |
| subjects in groups |
| \( s(g) \times \text{linear} \) |
| \( s(g) \times \text{quadratic} \) |
| \( s(g) \times (n-1) \text{th time} \) |
All three are symmetric matrices, and they contain all of the information that is necessary either for extracting the univariate repeated measures ANOVA results (if sphericity is satisfied) or for the more general MANOVA tests.

### 3.3.1 Extracting Univariate Repeated Measures ANOVA Results

Given the results of the previous section, it is a simple matter to extract the univariate ANOVA results, as shown in the following.

<table>
<thead>
<tr>
<th>Source</th>
<th>Source</th>
<th>SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS Time</td>
<td></td>
<td>$s_{t1} + s_{t2} + \cdots + s_{tn-1}$</td>
</tr>
<tr>
<td>SS Group</td>
<td></td>
<td>$s_{g0}$</td>
</tr>
<tr>
<td>SS Group $\times$ Time</td>
<td></td>
<td>$s_{g1} + s_{g2} + \cdots + s_{gn-1}$</td>
</tr>
<tr>
<td>SS Subjects within Groups</td>
<td></td>
<td>$s_{r0}$</td>
</tr>
<tr>
<td>SS Residual</td>
<td></td>
<td>$s_{r1} + s_{r2} + \cdots + s_{rn-1}$</td>
</tr>
</tbody>
</table>

The $F$-tests are then obtained as

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>MS</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$n-1$</td>
<td>$MST = \frac{SS_T}{(n-1)}$</td>
<td>$MST / MS_R$</td>
</tr>
<tr>
<td>Group</td>
<td>$s-1$</td>
<td>$MS_G = \frac{SS_G}{(n-1)}$</td>
<td>$MST / MS_{S(G)}$</td>
</tr>
<tr>
<td>Group $\times$ Time</td>
<td>$(s-1)(n-1)$</td>
<td>$MS_{GT} = \frac{SS_{GT}}{(s-1)(n-1)}$</td>
<td>$MS_{GT} / MS_R$</td>
</tr>
<tr>
<td>Subjects within Groups</td>
<td>$N-s$</td>
<td>$MS_{S(G)} = \frac{SS_{S(G)}}{(N-s)}$</td>
<td>$MS_{S(G)} / MS_R$</td>
</tr>
<tr>
<td>Residual</td>
<td>$(N-s)(n-1)$</td>
<td>$MS_R = \frac{SS_R}{(N-s)(n-1)}$</td>
<td></td>
</tr>
</tbody>
</table>

Testing of the individual trend components (i.e., linear, quadratic, etc.) uses $MS_R$ as the denominator error term as in the one-sample case. This is also the case for the individual group by time trend components (i.e., group by linear, group by quadratic, etc.). Again, the tests of the time and group by time interaction, as well as the tests for the individual trend components within both, are termed averaged tests because they utilize an averaged estimate of error (i.e., the common $MS_R$), which is appropriate only if sphericity holds.

### 3.3.2 Multivariate Tests

For the multivariate model, the overall group test is the same as in the univariate model. However, testing for the overall time and group by time effects both involve multivariate tests. The multivariate test of the group by time interaction is obtained by extracting the
(n - 1) \times (n - 1) submatrices of SSG* and SSR*, and solving the determinantal equation for min(s - 1, n - 1) latent roots:

\[ |SSG_{(n-1)}^* - \lambda SSR_{(n-1)}^*| = 0. \]  

(3.11)

Several test statistics are generally provided by standard statistical software for MANOVA: Roy's largest root statistic, Wilk's Lambda, Hotelling-Lawley Trace, and Pillai's Trace. Roy's largest root statistic is given by the first latent root or eigenvalue \( \lambda_1 \). Wilk's Lambda is computed as

\[ \Lambda = \prod_{h=1}^{s-1} \frac{1}{1 + \lambda_h}. \]  

(3.12)

Note that these same tests can be constructed for \( n - q - 1 \) terms if only a \( q < n \) degree trend is considered (e.g., only test for group by linear and quadratic trends even if \( n > 3 \)). To do this, one extracts and compares the corresponding \( n - q - 1 \) submatrices of the SSG* and SSR* matrices.

If the overall group by time test is nonsignificant, we may want to pool the interaction into the residual for (multivariate) testing of the time effect. Unfortunately, despite the increase in statistical power, pooling is not easily accomplished with most MANOVA statistical software packages. Instead, the usual multivariate test of the time effect involves the same determinantal equation as in the one-sample case:

\[ |SST_{(n-1)}^* - \lambda SSR_{(n-1)}^*| = 0. \]  

(3.13)

For both the group by time and time effects, testing for the significance of specific time-related components involves using a separate denominator for each, rather than the pooled MS_R that is used in the univariate model. For example, after the overall multivariate test of the group by time interaction, individual components are tested as

\[ F_{GT_1} = \frac{SSG_{(n-1)}^*/(s-1)}{SSR_{(n-1)}^*/(N-s)} \]  

\[ F_{GT_2} = \frac{SSG_{(n-1)}^*/(s-1)}{SSR_{(n-1)}^*/(N-s)} \]  

\[ \ldots \]  

(3.14)

each on \( s - 1 \) and \( N - s \) degrees of freedom.

3.4 ILLUSTRATION

To illustrate the multivariate approach, we return to the vocabulary growth data previously presented in Chapter 2 and in Bock [1975]. Summary statistics (vocabulary score means, standard deviations, and correlations between the grades) are presented in Table 2.1, and
are displayed graphically in Figure 2.1. Pre-multiplying the vector mean

\[
\tilde{y} = \begin{bmatrix}
1.14 \\
2.54 \\
2.99 \\
3.47
\end{bmatrix}
\]

by the orthogonal polynomial matrix

\[
P = \begin{bmatrix}
.5 & .5 & .5 & .5 \\
-.67082 & -.22361 & .22361 & .67082 \\
.5 & -.5 & -.5 & .5 \\
-.22361 & .67082 & -.67082 & .22361
\end{bmatrix},
\]

yields the following orthogonal polynomial estimates.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>5.0694</td>
</tr>
<tr>
<td>Linear</td>
<td>1.6658</td>
</tr>
<tr>
<td>Quadratic</td>
<td>-0.4606</td>
</tr>
<tr>
<td>Cubic</td>
<td>0.2224</td>
</tr>
</tbody>
</table>

The polynomial SST* matrix, presented in Table 3.1, is obtained by multiplying the squares and cross products of these coefficients by the number of subjects (\(N = 64\)). Similarly, the residual SSR* matrix in Table 3.1 is obtained by the matrix calculation:

\[
SSR^* = P(Y'Y - N\tilde{y}\tilde{y})P'.
\]

As noted, these are referred to as sum of squares and cross-product (SSCP) matrices. In the present case, the SSR* matrix conforms to mixed-model assumptions in that the off-diagonal elements (of the lower three-by-three portion of this matrix) are small relative to their corresponding diagonal elements, and the last three diagonal elements are of the same magnitude. A test of this assumption (not shown) does not reject sphericity. However, for illustration, we will interpret the multivariate results for these data and compare them to the univariate results presented in Chapter 2.

To obtain the multivariate test for the grade effect, the determinantal equation

\[
|E^{-1} SST^*_{(n-1)}(E^{-1})' - \lambda I_{(n-1)}| = 0,
\]

using the Cholesky factorization SSR^*_{(n-1)} = EE', yields 4.7432 as Roy’s largest root statistic (eigenvalue or latent root \(\lambda_i\)). Similarly, Wilks’ Lambda equals \(1/(1 + 4.7432) = .17412\). It can be shown that these translate to a \(F\)-value of 96.45 (see Finn [1974]) with \(df = 3, 61\), which is highly significant: \(p < .0001\). Thus, there is clearly a grade effect on average vocabulary scores.
In terms of the individual trend components, results of the analysis are quite similar to
the univariate results presented in Chapter 2, though the cubic term is even less significant
here. As in the univariate analysis, both the linear and quadratic trend components are
highly significant, and the above polynomial contrast estimates indicate a positive linear
and negative quadratic effect. Vocabularily growth is increasing across these grades, but
at a decelerating rate. As mentioned, the multivariate analysis uses separate denominators
for forming these $F$-values for the trend components. For example, under the multivariate
model the $F$-test for the linear trend component is calculated as

\[
\text{Linear } F = \frac{177.59}{(50.42/63)} = 221.88,
\]

which is compared to the $F$-distribution with 1 numerator and 63 denominator degrees of
freedom. Alternatively, under the univariate repeated measures ANOVA, this same test
uses the pooled $MS_R$ to yield

\[
\text{Linear } F = \frac{177.59}{(((50.42 + 43.95 + 60.57)/3)/63)} = 216.63,
\]

which is compared to $F$ with 1 numerator and 189 denominator degrees of freedom. Both
are highly significant in this case, but the point is that the critical value is smaller under the
univariate model assumptions because of the pooling of the error term.

The univariate sums of squares in Tables 2.2 and 2.3, may be obtained from Table
3.1 as follows: (a) The constant term is the first diagonal element of $SST^*$, and the linear,
quadratic, and cubic polynomial grade sums of squares are the remaining diagonal elements
of $SST^*$; (b) the subject sums of squares is the first diagonal element of $SSR^*$, and (c) the
residual sums of squares is the sum of the remaining three diagonal elements of $SSR^*$.

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SSCP</th>
<th>$F$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time SST*</td>
<td>1</td>
<td>1644.71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td></td>
<td>177.59</td>
<td>221.88</td>
<td>.0001</td>
</tr>
<tr>
<td>Linear</td>
<td></td>
<td>540.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadratic</td>
<td></td>
<td>-149.45</td>
<td>13.58</td>
<td>.0001</td>
</tr>
<tr>
<td>Cubic</td>
<td></td>
<td>72.16</td>
<td>-6.56</td>
<td>.075</td>
</tr>
<tr>
<td>Residual SSR*</td>
<td>63</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Between subjects</td>
<td></td>
<td>873.60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Linear error</td>
<td></td>
<td>3.84</td>
<td>50.42</td>
<td></td>
</tr>
<tr>
<td>Quadratic error</td>
<td></td>
<td>-49.82</td>
<td>43.95</td>
<td></td>
</tr>
<tr>
<td>Cubic error</td>
<td></td>
<td>-23.76</td>
<td>-4.27</td>
<td>60.57</td>
</tr>
</tbody>
</table>
3.5 SUMMARY

In summary, ANOVA and MANOVA approaches for analysis of longitudinal data represent well-understood and well-developed statistical methodologies. In addition, there is considerable available computer software for their computation. The results are based on relatively simple and noniterative calculations. Both models, unfortunately, have features which limit their usage in longitudinal data analysis. The ANOVA model for repeated measurements assumes sphericity, which is unrealistic for many applications where variances tend to increase with time and correlation decreases with increasing intervals in time. Alternatively, while the MANOVA model allows for a general variance-covariance structure for the repeated measures, it has the disadvantage of requiring complete data for all subjects and identical measurement occasions. Unfortunately, this overly stringent requirement is violated in most cases. Furthermore, software implementations of the multivariate model often provide only limited ways of handling covariates. In the following chapters, we consider more general models that overcome the limitations of these traditional approaches.